The separation of a coupled system of differential equations in quantum mechanics

1992 J. Phys. A: Math. Gen. 25 3749

(http://iopscience.iop.org/0305-4470/25/13/023)

J. Phys. A: Math. Gen. 25 (1992) 3749-3761. Printed in the UK

The separation of a coupled system of differential equations in quantum mechanics

Cao xuan Chuan†

Observatoire de Nice, BP 35, Nice, France

Received 30 October, in final form 9 March 1992

Abstract. We discuss various aspects of the methodology of the separation operation of a coupled system of differential equations in quantum mechanics, namely the Darboux' transformation, the T and N matrices approach for the case of symmetric and non-symmetric coupling of the equations. The analysis is also extended to systems with more than two equations with, in each case, specific examples illustrating their implementation in practical situations.

1. Theory

Consider the following coupled system of linear differential equations which, in matrix form, can be written as

$$[PI + D]\phi = \lambda\phi \tag{1}$$

where n is the number of equations, I is the unit $n\times n$ matrix, $\phi = (\phi_1, \ldots, \phi_n)$, P is a linear differential operator and D a non-diagonal $n\times n$ matrix, $D = (u_i, d_{ij})$ $i, j = 1, 2, \ldots, n$ and λ a constant diagonal matrix $\lambda = (\lambda_1, \ldots, \lambda_m)$, $d_{ij} \neq d_{ji}$ (non-symmetrical coupling).

The separation operation of system (1) is a transformation N such that the original base ϕ is transformed into a new one $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$

$$\psi = N\phi \tag{2}$$

which verifies the separated equations in the system

$$[P + F]\psi = \lambda\psi \tag{3}$$

where F is now a $n\times n$ diagonal matrix $F = (v_1, v_2, \ldots, v_n)$ with n unknown elements v_i. If these elements v_i can be determined and if the separated equations can be solved, then the original solutions ϕ_i can be recovered by an inverse transformation

$$\phi = N^{-1}\psi. \tag{4}$$

The sepration operation is, however, not always possible unless some constraints on the elements of D are imposed, and the study of these constraints constitutes an interesting trend of research in mathematical physics.

It has already been shown that, in the case of a system of two coupled equations with symmetric coupling and resonance condition ($\lambda_1 = \lambda_2$), complete separation can be obtained through the 'T' transformation approach if these elements are linked by a simple relation (Cao 1981).

† Permanent address: 01 Parvis du Breuil, 92160 Antony, France.

0305-4470/92/133749+13$04.50

It was pointed out later (Humi Mayer 1985) that this constraint may eventually be enlarged by use of another approach based on the Darboux's transformation, a prototype of the Lie–Backlund transformation (Lamb 1980) in which the operator P must be of second order ($P = \mathrm{d}^2/\mathrm{d}x^2$), that is to say the matrix N may have the form

$$N = A(x) + B(x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{5}$$

$A(x):(a_{ij})$, $B(x):(b_{ij})$ are the $n \times n$ non-diagonal matrices to be determined.

Initially, in attempting to apply it to some practical problems, we are led to a number of observations and remarks which will be first analysed. From this study, a number of new improvements and understanding of the methology can be extracted, with more insight into the connection between the Darboux and the T approach. A new type of transformation, the N matrix, will next be introduced to solve the problem of non-symmetrical couplings. Both the T and N approaches are governed by the theorem of separation and are useful in quantum mechanics as can be seen in the following examples. Finally we also show that their combination may provide a convenient means to solve the problem of three coupled equations.

1.1. The Darboux's transformation

From (5), using $P = \mathrm{d}^2/\mathrm{d}x^2$, we may write

$$\psi = A\phi + B\phi'. \tag{6}$$

Replacing this in (1) we can infer the two following matrix equations:

$$\begin{aligned} &-A'' - 2\lambda B' + 2B'D + BD' + AD = FA \\ &-2A' + BD - B'' = FB \end{aligned} \tag{7}$$

where the matrices $A(x)$ and $B(x)$ are in principle arbitrary but we note that for quantum mechanics, the most significant choices are the cases $B(x) = 1$ and $B(x) = 0$.

Case $B(x) = 1$. For simplicity we consider the case $n = 2$ (two coupled equations) so that

$$-A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} u_1 & d_1 \\ d_2 & u_2 \end{pmatrix}$$

$$F = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \qquad \lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_l \end{pmatrix}.$$

From the second equation in (7) we have

$$a'_{12} = -\tfrac{1}{2}d_1 \qquad a'_{21} = -\tfrac{1}{2}d_2.$$

Let $c_i = \int^x d_i(x)\,\mathrm{d}x$ $i = 1, 2$ then, neglecting the constant of integration

$$a_{12} = -\tfrac{1}{2}c_1 \qquad a_{21} = -\tfrac{1}{2}c_2.$$

Combining these results with (7), it can be verified that the elements a_{ii} must simultaneously satisfy the following relations:

$$a_{11} = \frac{1}{c}[\alpha - \tfrac{1}{2}d_1 + I_1] \qquad a_{22} = \frac{1}{c}[-\alpha - \tfrac{1}{2}d_2 - I_2] \tag{8a}$$

where α is an arbitrary constant of integration.

$$I_i = \tfrac{1}{2}\int_x^x c_i(u_2 - u_1)\,\mathrm{d}x + \tfrac{1}{2}(\lambda_2 - \lambda_1)\int^x c_i\,\mathrm{d}x$$

$$a'_{11} - a^2_{11} = \tfrac{1}{2}\int^x c_1 c'_2\,\mathrm{d}x - u_1 \tag{8b}$$

$$a'_{22} - a^2_{22} = \tfrac{1}{2}\int^x c_2 c'_1\,\mathrm{d}x - u_2.$$

It can be seen immediately that no simple solution can be reached in the general case unless a first constraint $d_1 = d_2 = d$ (symmetric coupling) is admitted that is to say

$$c_1 = c_2 = c \qquad I_1 = I_2 = I.$$

Writing now explicitly the four relations in (8), we have

$$a_{11} = \frac{1}{c}[\alpha - \tfrac{1}{2}d + I]$$

$$a_{22} = \frac{1}{c}[-\alpha - \tfrac{1}{2}d - I] \tag{9a}$$

$$a^1_{11} - a^2_{11} = \tfrac{1}{4}c^2 - u_1$$

$$a^1_{22} - a^2_{22} = \tfrac{1}{4}c^2 - u_2. \tag{9b}$$

The compatibility between (9*a*) and (9*b*) requires a second type of constraint which can be formulated as follows: if the difference $u_1 = u_2$ is given *a priori*, then the sum $u_1 + u_2$ must comply with the relation

$$u_1 = u_2 = \frac{1}{2}c^2 + \left(\frac{d}{c}\right)' + \frac{1}{2}\left(\frac{d}{c}\right)^2 + \frac{2}{c^2}(\alpha + I)^2. \tag{10}$$

Note that if $\lambda_1 = \lambda_2 = \lambda$ (resonance condition), and replacing c by $2c$, relation (10) exactly agrees with the result first obtained by Mayer (1985). The elements v_1, v_l of the diagonal matrix F are

$$v_1 = u_1 + 2a'_{11}$$

$$v_2 = u_2 + 2a'_{22}. \tag{11}$$

Hence, if the separated equations (3) can be solved, the original solution ϕ can be recovered by an inverse transformation

$$\phi = [A^{-1}C]^{-1}(\psi - A^{-1}\psi')$$

$$C = A' + D - A$$

provided that the matrices A and C are not singular everywhere. Therefore a third kind of constraint must be imposed:

$$\Delta(A) \neq 0 \qquad \Delta(C) \neq 0 \tag{12}$$

in which the symbol Δ corresponds to the determinant of the corresponding matrix.

Example 1. As an illustration of the above considerations, we choose the case previously given in Mayer (1985), in which

$$c = ax^k \qquad u_2 - u_1 = bx^l \qquad \lambda_1 = \lambda_2 = \lambda \tag{13}$$

where k, l are parameters, a, b are arbitrary constants. If the constant of integration α is set equal to zero, the expression for $\Delta(A)$ is

$$\Delta(A) = \frac{1}{4}\frac{k^2}{x^2} - \left[a^2 x^{2k} + \frac{b^2 x^{2(l+1)}}{(k+l+1)^2}\right]. \tag{14}$$

Note that with increasing values of x the first term of the RHS is always decreasing while the second term is always increasing. This means that there exits at least one (or more) roots r_0 such that $\Delta(A(r_0)) = 0$ so that the constraint in (12) cannot be met and this case seems not fit for the problem under consideration, even when separation of the coupled equations is possible. These difficulties become dramatically transparent when we extend the analysis to the case $n > 2$. For instance for $n = 3$, we find that the determination of the elements a_{ii} must be subject to the compatibility requirements of three kinds of differential equations; two are linear but the third is nonlinear (Ricatti's type) excluding any attempt to extract simple analytic relation such as (10). However, when a physical situation does not require the reconstruction operation, then the $B = I$ approach may become useful, but this constitutes a different topic beyond the scope of the present paper and will be discussed later on. It was also pointed out in a subsequent paper (Mayer 1987) that the transformation corresponding to $B = 1$ is not invertible except for the simple case $n = 1$.)

Case $B = 0$. In order to avoid these difficulties, the case $B = 0$ is more promising not only because of its relative simplcity but also because it is susceptible to generalization with a broader range of utilization. In fact, from the second equation in (7), the matrix A is now a constant matrix so that the transformation N is identical to A and is determined by

$$AD = FA \tag{15}$$

the constant matrix A must be non-singular, making possible the reconstruction operation as well as normalization of the original functions ϕ_i. The dfficulty here lies in the fact that with $n = 2$ (resonance and symmetric coupling) there are in principle six unknowns (a_{ij}, v_i) while with (15) we have at our disposal only four equations. Some constraints must therefore be set up among the elements of the matrix D; they are precisely governed by the theorem of separation (Cao 1981); that is to say $(u_1 - u_2)^{-1} d$ = constant and the appropriate transformation N become identical to the transformation $T(a)$

$$T(a) = \begin{pmatrix} 1-a & 1+a \\ -(1+a) & 1-a \end{pmatrix} \tag{16}$$

with

$$a = -2\gamma \pm \sqrt{1+4\gamma^2} \qquad \gamma = (a_1 - a_2)^{-1} d.$$

Note also that if this constraint cannot be fully satisfied, it is generally possible, using the technique of the auxiliary parameters (Cao 1982, 1988), to seek for a partial separation such that the problem to amenable to a perturbation treatment. Some details on numerical aspects applied to quantum mechanics can be found in Bougouffa *et al* (1988, 1989).

However, in the present work, we shall limit ourselves to the case of exact separation in order to see how this can be extended to more general situations. One of the advantages of the $T(a)$ transformation method lies in the fact that the differential operator P need not be limited to the second order; we may have for instance

$$P=\sum a_m \frac{\mathrm{d}^m}{\mathrm{d}x^m} \qquad m: 1, 2 \ldots \qquad a_m: \text{constant}$$

On the other hand $T(a)$ can also be considered as representations of a Lie group of dimensionality unity with generator

$$I_1=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and with the unit element corresponding to $\gamma=0$ (or $a=\pm 1$). Two successive transformations $T(a_1)$, $T(a_2)$ yield another transformation $T(a)$ where

$$a=\frac{a_1+a_2-a_1a_2+1}{a_1+a_2+a_1a_2-1}.$$

Furthermore, we have found that it is in fact possible to go beyond the linear constraint or the symmetric coupling case. As, to our knowledge, this extension of the theorem of separation has not yet been clarified in current literature, we shall proceed with some details in the discussion below.

1.2. The 'differential constraint'

Let $z=u_1-u_2$ and consider the symmetric coupling $(d_1=d_2=d)$ in (1). Assume that d is related to z by the constraint

$$d=z^m \frac{\mathrm{d}^n}{\mathrm{d}x^n} z^r$$

in which m, n, r are parameters. With the same method it can then be shown that system (1) can also be completely separated if the quantity z is a solution of the following equation:

$$z^m \frac{\mathrm{d}^n}{\mathrm{d}x^n}(z^r)-\frac{1-a^2}{4a} z=0$$

in which a is now considered as a parameter. Obviously, this equation can be solved analytically for specific values of the parameters m, n, r. For example if $n=1$, $r=1$, the case $m=-1$ corresponds to a Ricatti, while $m=-2, -3, \ldots$ corresponds to the Bernoulli equations. We may also extend the constraint to the form $d=x\varphi(z')+\psi(z')$, φ and ψ are arbitrary functions of z' then (17) is reduced to the Lagrange or Clairault's type of equations.

Example. Returning to the case (13) it can be verified that if we take $l=\frac{1}{2}$, $c=\frac{1}{3}b^3x^{3/2}$, $m=2$, $n=1$, $r=1$ the system (1) is completely separated with the parameter a given by $a=-b^2\pm\sqrt{1+b^4}$. The transformation $T(a)$ is now invertible and the diagonal elements of $\bar{D}$ are simply

$$v_{\substack{1\\2}}=\tfrac{1}{2}[u_1+u_2\mp g(\alpha)(u_1-u_2)]$$

$$g(a)=\frac{1+a^2}{2a}$$

1.3. Non-symmetric coupling

We consider system (1) in which the matrix D has the form

$$D=\begin{pmatrix} u_1 & d_1 \\ d_2 & u_2 \end{pmatrix} \qquad d_1 \neq d_2 \tag{17}$$

and prove the following statement which, in a sense, is a mere extension of the theorem of separation mentioned above.

Statement. The system of two coupled differential equations in (1) with the matrix D given in (17) can always be completely separated if simultaneously $d_1(\Delta u)^{-1}=$ constant; $d_2(\Delta u)^{-1}=$ constant; $\Delta u = u_1 - u_2$.

Proof. Using the following transformation:

$$\psi = N(X, Y)\phi \qquad \psi=(\psi_1, \psi_2) \tag{18}$$

$$N(X, Y)=\begin{pmatrix} 1 & X \\ -Y & 1 \end{pmatrix} \tag{19}$$

where X, Y are unknowns for the moment, the new system is

$$[P+F]\psi = \lambda\psi \tag{20}$$

$$F=-\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}. \tag{21}$$

The elements X and Y are determined by two equations

$$d_2X^2 - \Delta u X - d_1 = 0 \qquad d_1 Y^2 - \Delta u Y - d_2 = 0 \tag{22}$$

so that

$$X=\frac{1}{2}\frac{\Delta u}{d_2} \pm \sqrt{\left(\frac{\Delta u}{2d_2}\right)^2 + \frac{d_1}{d_2}} \qquad Y=\frac{1}{2}\frac{\Delta u}{d_1} \pm \sqrt{\left(\frac{\Delta u}{2d_1}\right)^2 + \frac{d_1}{d_1}}.$$

The elements v_1, v_2 are then

$$v_1 = u_1 + d_1 Y \qquad v_2 = u_2 - d_2 X. \tag{23}$$

Remarks. (i) In the special case where P is a first-order differential operator ($P = \mathrm{d}/\mathrm{d}x$), the above linear or differential constraints become redundant. In this case the quantities X, Y are function of x and must be solutions of two nonlinear differential equations of the Ricatti type.

(ii) It can be verified that in the case of symmetric coupling ($d_1 = d_2 = d$), $X = Y$ and the expressions for v_1 and v_l are

$$v_1 = \tfrac{1}{2}(u_1+u_2) - \tfrac{1}{2}\Delta u\sqrt{1+4\gamma^2}$$
$$\gamma = (\Delta u)^{-1} d$$
$$v_2 = \tfrac{1}{2}(u_1+u_2) + \tfrac{1}{2}\Delta u\sqrt{1+4\gamma^2}$$

in which we may recognize the characteristic form of the results obtained with the $T(a)$ transformation that is to say in this case we have identity between $(1-a)N$ and T justifying therefore the consistency of its use $X=(1+a)/(1-r)$.

(iii) For the antisymmetric coupling case ($d = d_1 = -d_2$), note that $X_\pm = -Y_\mp$ and these quantities may be real or imaginary according to $\Delta u \gtreqless 2d$.

2. Applications

2.1. Examples

In order to see how this can be implemented, we shall consider two examples, the first one may be seen as a pedagogical exercise and the second one does correspond to a real physical situation.

Case 1. Let the matrices D and λ in (1) be defined by

$$u = u_1 = u_2 \qquad d_1 = d_2 = -d \qquad \lambda_1 \neq \lambda_2$$

We shall show that the solutions (ϕ_1, ϕ_2) can always be expressed in terms of two supersymmetric partners (ψ_1, ψ_2) corresponding to a superpotential v'. This can be proved in two steps:

(i) Note first that we are in the case of symmetric coupling implying the use of the T matrix approach. As $u_i = u_2$, we may take $a = 0$ in $T(a)$ and transform the base ϕ into $\bar{\phi}$ where $\bar{\phi} = T(0)\phi$

$$[P + \bar{D}]\bar{\phi} = \bar{\lambda}\bar{\phi} \tag{24}$$

in which

$$\bar{D} = \begin{pmatrix} u+d & 0 \\ 0 & u-d \end{pmatrix} \qquad \bar{\lambda} = \begin{pmatrix} \lambda_+ & \lambda_- \\ \lambda_- & \lambda_+ \end{pmatrix} \qquad \lambda_\pm = \tfrac{1}{2}(\lambda_1 \pm \lambda_2). \tag{25}$$

(ii) To unveil the supersymmetric character of relations (24), (25) we use the '$\hat{C}$' transformation technique (Cao 1991) in going into a new base ψ:

$$\bar{\phi} = \hat{C}\psi \qquad \hat{C} = c(x)I \qquad \psi = (\psi_1, \psi_2).$$

The explicit form of the resulting equations is

$$\begin{gathered} \left[\frac{\mathrm{d}}{\mathrm{d}x} + u + \frac{c'}{c} + d - \lambda_+\right]\psi_1 = \lambda_-\psi_2 \\ \left[-\frac{\mathrm{d}}{\mathrm{d}x} - \left(u + \frac{c'}{c}\right) + d + \lambda_+\right]\psi_2 = -\lambda_-\psi_1. \end{gathered} \tag{27}$$

As $c(x)$ is arbitrary, we may now choose it as $c(x) = K\,\mathrm{e}^{-B(x)+\lambda_+ x}$ K is a constant and $B(x) = \int^x u(x)\,\mathrm{d}x$. The above system becomes

$$\begin{gathered} \left[\frac{\mathrm{d}}{\mathrm{d}x} + d\right]\psi_1 = \lambda_-\psi_2 \\ \left[-\frac{\mathrm{d}}{\mathrm{d}x} + d\right]\psi_2 = -\lambda_-\psi_1 \end{gathered} \tag{28}$$

in which we may recognize the supersymmetric character corresponding to the usual 'ladder operator' and superpotential, $A^\pm$ and v', where

$$A^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}x} + v' \qquad v' = d$$

while the components ψ_1, ψ_2 are solutions of the equations

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + d^2 \mp d' + \lambda_-^2\right]\psi_{\substack{1\\2}} = 0 \tag{29}$$

and the original functions ϕ_1, ϕ_2 are related to ψ_1, ψ_2 by

$$\phi_{\substack{1\\2}} = \tfrac{1}{2}K\,\mathrm{e}^{-B(x)+\lambda_+ x}[\psi_1 \mp \psi_2].$$

Conditions for 'good symmetry' will in turn impose further choices of the functions $u(x)$, $d(x)$.

Case 2. We take now a real physical situation concerning the Dirac equation with an attractive Coulomb potential discussed in Sukumar (1985), where the matrix D of equation (2) of this reference should be

$$D = \frac{1}{2}\begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix} \tag{30}$$

and where

$$\lambda = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \qquad \alpha_{\substack{1\\2}} = m \pm E$$

k is an eigenvalue of the operator $-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ and γ, the strength of the Coulomb interaction ($\gamma = \alpha Z_1$; α is a fine structure constant, Z is the charge number). We are now in the presence of antisymmetric couplings which requires the use of the $N(X, Y)$ approach recalling that $X_\pm = -Y_\mp$. After some simple algebra and noting that $X_+ = -Y_- = \gamma^{-1}(k+s)$ in which we set $s = \sqrt{k^2-\gamma^2}$ and using new coordinates defined by $\rho = Ex$ the system (1) can be written explicitly as:

$$\begin{gathered} \left[\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right]\psi_2 = \left[\frac{k}{s} + \frac{m}{E}\right]\psi_1 \\ \left[-\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right]\psi_1 = \left[\frac{k}{s} - \frac{m}{E}\right]\psi_2. \end{gathered} \tag{31}$$

An equivalent result can also be obtained if we use the special transformation

$$M = \begin{pmatrix} k+s & -\gamma \\ -\gamma & k+s \end{pmatrix}. \tag{32}$$

In fact, the relation between $N(X, Y)$ and M is simply

$$N = -\frac{1}{\gamma}M\sigma_1 \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

Note that the quantity s may be real or imaginary depending on whether $\gamma > k$ or $\gamma < k$ (for example low lying states of heavy hydrogen-like atoms $Z > 70$). The ladder operator $A^\pm$ and superpotential v' are

$$A^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}x} + v' \qquad v' = \frac{s}{\rho} - \frac{\gamma}{s}. \tag{34}$$

Some physical aspects of this description have already been discussed in Sukumar (1985) (see also Ui 1984).

It is perhaps worthwhile to mention that while the M transformation is a special type pertaining to the case of the Dirac equation, the $N(X, Y)$ transformation has a more general range of use.

Remark. It may also be enlarged when combined with the '$\hat{C}$' construction technique outlined in example 1. For instance, if we start from (31) and use the '$\hat{C}$' transformation to define a new base $\bar{\psi}=(\bar{\psi}_1, \bar{\psi}_2)$

$$\bar{\psi}=\hat{C}\psi$$

the new system is

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}x}+v_2'\right]\bar{\psi}_2&=\alpha_+\bar{\psi}_1\\ \left[-\frac{\mathrm{d}}{\mathrm{d}x}+v_1'\right]\bar{\psi}_1&=\alpha_-\bar{\psi}_2\end{aligned} \qquad (35)$$

with the following notation

$$\begin{aligned}&v'_{\substack{1\\2}}=\frac{s}{\rho}-\frac{\gamma}{s}\mp v'\\ &\hat{C}=c(x)I \qquad \alpha_\pm=\frac{k}{s}\pm\frac{m}{E} \qquad \frac{c'}{c}=v'\end{aligned} \qquad (36)$$

where I is the unit matrix. The function $c(x)$ is still arbitrary and can be chosen conveniently. For example if we take

$$c(x)=\exp\left[-\frac{1}{2^{3/2}}mx^2\right]$$

so that

$$\frac{c'}{c}=-\frac{1}{2^{1/2}}mx$$

where m is mass, then the above system become physically meaningful ($x=\rho E$):

- For $\gamma=0$ it can represent the case of a Dirac equation in which linearity of the coordinate is assumed or, in other words, it can be associated (with some slight modifications) to the problem of the 'Dirac oscillator' discussed in Moshinsky (1989) and Quesne (1991).
- If $\gamma\neq 0$, this system represents then a new situation in which are simultaneously present the Coulomb and oscillator interaction. the presence of the oscillator interaction destroys the symmetry so that the components $\bar{\psi}_1$, $\bar{\psi}_2$ are no more SUSY partners because the 'ladder operators'

$$A_1^-=-\frac{\mathrm{d}}{\mathrm{d}r}+v_1' \qquad A_2^+=\frac{\mathrm{d}}{\mathrm{d}x}+v_2'$$

are not adjoint.

Nevertheless, the concept of supersymmetry can be retained in the following sense: consider now two operators

$$A_1^+=\frac{\mathrm{d}}{\mathrm{d}x}+v_1' \qquad A_2^-=-\frac{\mathrm{d}}{\mathrm{d}x}+v_2'$$

so that $A_i^\mp$ are adjoint. The SUSY partners $(\bar{\psi}_{1,s}, \bar{\psi}_{2,s})$ of $(\bar{\psi}_1, \bar{\psi}_2)$ are given by

$$\begin{aligned}&A_2^-\bar{\psi}_2=\alpha_+\bar{\psi}_{2,s} \qquad A_2^-\bar{\psi}_{2,s}=\alpha_-\bar{\psi}_2\\ &A_1^-\bar{\psi}_1=\alpha_-\bar{\psi}_{1,s} \qquad A_1^+\bar{\psi}_{1,s}=\alpha_+\bar{\psi}_1.\end{aligned} \qquad (37)$$

Therefore, the concept of a superpotential according to Witten (1981) must now be replaced by a 'matrix superpotential' (Amado *et al* 1988) w' defined by

$$w'=\begin{pmatrix} v_1' & 0 \\ 0 & v_2' \end{pmatrix}. \tag{38}$$

The 'charge operator' is now a 4×4 matrix

$$Q^+=\begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \qquad Q^-=\begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}$$

with

$$A^+=\begin{pmatrix} A_1^+ & 0 \\ 0 & A_2^+ \end{pmatrix} \qquad A^-=\begin{pmatrix} A_1^- & 0 \\ 0 & A_2^- \end{pmatrix}$$

as well as the Hamiltonian

$$H=\begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} \qquad H^-=\begin{pmatrix} A_1^-A_1^+ & 0 \\ 0 & A_2^-A_2^+ \end{pmatrix} \qquad H^+=\begin{pmatrix} A_1^+A_1^- & 0 \\ 0 & A_2^+A_2^- \end{pmatrix}. \tag{39}$$

By construction we always have nilpotency $(Q^{\pm})^2=0$ and it can be verified that

$$\{Q^+, Q^-\}=2H \qquad [Q^{\pm}, H]=0$$

the notation [,], {,} representing the commutator and anticommutator. This point of view will be developed separately in a more general context.

3. Extension

3.1. System with more than two coupled equations

This is a system (1) in which the matrix D is now a $n\times n$ matrix $D=(u_i, d_{ij})$ $i, j=2, \ldots, n$ $n>2$, $\phi=(\phi_1, \phi_2, \ldots, \phi_n)$. Note first that literature pertaining to such a system used to describe a given physical situaton remains scarce. In fact, complete separation is generally not possible even when a set of linear constraints is assumed. However, a number of interesting cases can be formulated by the two following statements, and are useful for further work:

(a) It is always possible to transform the symmetric $n\times n$ matrix D into a triangular matrix $\bar{D}$ such that the system (1) can be solved successively by a substitution method if only a single linear constraint is assumed.

Proof. Let D be symmetric and $n=3$

$$D=\begin{pmatrix} u_1 & d_1 & d_2 \\ c_1 & u_2 & d_3 \\ d_2 & d_3 & u_3 \end{pmatrix}.$$

Assume the linear constraint $(u_1-u_2)^{-1}d_1=\gamma_1=\text{constant}$, d_2, d_3, u_3 may be any analytic functions, $\lambda_1=\lambda_2\neq\lambda_3$.

Consider the generalized $T_3(a_1)$ transformation

$$T_3(a_1)=\begin{pmatrix} T_2(a_1) & 0 \\ 0 & 1 \end{pmatrix}. \tag{40}$$

$T_2(a_1)$ is given in (16) and $a_1=-2\gamma_1\pm(1+4\gamma_1^2)^{1/2}$. Repeating twice the transformation $T_3(a_1)$ on system (I) provides the following new system:

$$[P+D^{(2)}]\phi^{(2)}=\lambda\phi^{(2)} \tag{41}$$

in which $\phi^{(2)}=(\phi_1^{(2)},\phi_2^{(2)},\phi_3^{(2)})$

$$D^{(2)}=\begin{pmatrix} u_1^{(2)} & d_1^{(2)} & (1/\Delta_1^2)d_2^{(2)} \\ d_1^{(2)} & u_2^{(2)} & (1/\Delta_1^2)d_3^{(2)} \\ d_2^{(2)} & d_3^{(2)} & u_3 \end{pmatrix}$$

$$\Delta_1=2(1+a_1^2) \qquad u_3^{(2)}=u_3$$

$$u_{\substack{1\\2}}^{(2)}=\tfrac{1}{2}(u_1^{(1)}+u_2^{(1)})\mp\frac{a_1}{1+a_1^2}(u_1^{(2)}-u_2^{(1)}) \qquad d_2^{(2)}=d_2^{(1)}(1-a_1)-d_3^{(1)}(1+a_1)$$

$$u_{\substack{1\\2}}^{(1)}=\tfrac{1}{2}(u_1+u_2)\mp\tfrac{1}{2}(u_1-u_2)\sqrt{1+4\gamma_1^2} \qquad d_3^{(2)}=d_2^{(1)}(1+a_1)+d_3^{(1)}(1-a_1) \tag{42}$$

$$d_1^{(2)}=\frac{1}{2}\frac{1-a_1^2}{1+a_1^2}(u_1^{(1)}-u_2^{(1)}) \qquad d_1^{(2)}(u_1^{(2)}-u_2^{(2)})=\text{constant}$$

and the linear constraint is always conserved. Repeating then the operation once more with another appropriate parameter a_l etc and after $2m$ such transformation, the structure of the resulting $D^{(2m)}$ matrix is

$$D^{(2m)}=\begin{pmatrix} u_1^{(2m)} & d_1^{(2m)} & (1/\Delta)d_2^{(2m)} \\ d_1^{(2m)} & u_2^{(2m)} & (1/\Delta)d_3^{(2m)} \\ d_2^{(2m)} & d_3^{(2m)} & u_3 \end{pmatrix} \tag{43}$$

where

$$\Delta=\prod_{i=1}^{m}\Delta_i^2.$$

If the parameters a_i are real, we have $\Delta_i>1$ so that it will always be possible to choose m such that $\Delta\gg1$ and neglect the quantities $d_2^{(2m)}\Delta^{-1}$, $d_3^{(2m)}\Delta^{-1}$ i.e. to disconnect the subset $(\phi_1^{(2m)},\phi_2^{(2m)})$ from $\phi_3^{(2m)}$. In this subset, the equations are still coupled but from conservation of the linear constraint we may use a supplementary transformation $T_2(a_{m+1})$ to separate them and solve the resulting equations. The results will then be substituted in the third equation to infer $\phi_3^{(2m)}$. Obviously, this method can be extended to system of $n>3$ equations by use of the chain of transformations $T_m, T_{m-1},\ldots,T_2$. It also completes the results presented in an earlier paper (Cao 1988).

Complete separation would require the more severe constraints

$$\frac{d_{ij}}{u_k}=\text{constant} \qquad i\neq j\neq k \qquad (i,j,k=1,\ldots,n) \tag{44}$$

and $\lambda_i=\lambda$. This means that the matrix D can be written as $D=f(x)A$ in which $f(x)$ may be any analytic function and A an $n\times n$ constant matrix. In principle, it can be diagonalized by the conventional method of eigenvalues, which requires the solution of an algebraic equation of nth order of the form

$$\sum_{i=1}^{n}a_ix^i=0.$$

For instance if $n=3$ and $f(x)$ is proportional to x^{-2}, the system (I) has been used to analyse the $|n, l\rangle \to |n, l\pm 1\rangle$ optical transition in the $e-H$ collision according to the Lane–Lin model (Lane and Lin 1964), which made use of a result originally obtained by Seaton (1961) in which one of the coupling terms is neglected ($d_2=0$) by physical arguments.

As an extension, we find that the T and N transformations constitute an interesting alternative approach with a notable advantage because it enables us to replace the nth order algebraic equation by a set of $n-1$ second-order equations which are simpler to handle. In order to illustrate this point, we consider again the case $n=3$, and introduce the $N_3(X, Y)$ matrix

$$N_3(X, Y) = \begin{pmatrix} 1 & 0 \\ 0 & N_2(X, Y) \end{pmatrix} \tag{45}$$

$N_2(X, Y)$ already given in (19), and prove the following statement:

(b) Let D be symmetric satisfying (44); if only one of the following constraints is assumed

$$-\frac{d_k - d_j}{d_k + d_j} = a_i \qquad i \neq j \neq k \tag{46}$$

then system (1) can always be separated completely.

Proof. We shall proceed in two steps:

(1) With the transformation $T_3(a)$ the matrix D become $\bar{D}(\bar{u}_i, \bar{d}_j)$

$$\bar{D} = \begin{pmatrix} \bar{u}_1 & 0 & 0 \\ 0 & \bar{u}_2 & (1/\Delta_1)d_3 \\ 0 & \bar{d}_3 & u_3 \end{pmatrix}$$

where

$$\Delta_1 = 2(1+a_1^2) \qquad \bar{d}_3 = \frac{\Delta_1}{1-a_1} d_3 \qquad \bar{u}_{\substack{1\\2}} = \tfrac{1}{2}(u_1+u_2) \mp \tfrac{1}{2}(u_1-u_2)\sqrt{1+4\gamma_1^2}.$$

(2) Noting that $\bar{D}$ is non-symmetric, we must apply the transformation $N_3(X, Y)$ in which X and Y are solutions of two second-order equations similar to (22). The resulting matrix $\bar{\bar{D}}(v_i)$ is now diagonal. We find

$$\begin{aligned} v_1 &= \bar{u}_1 \\ v_2 &= \bar{u}_2 - \bar{d}_3 b \\ \bar{v}_3 &= \bar{u}_3 + \bar{d}_3 b \\ b &= \frac{1}{2}\left[R \pm \left(R^2 + 4\frac{\Delta_1}{(1-a_1^2)^2} d_3^2\right)^{1/2}\right] \\ R &= \tfrac{1}{2}[u_1 + u_2 - 2u_3 \pm (u_1 - u_2)\sqrt{1+4\gamma_1^2}]. \end{aligned} \tag{47}$$

To summarize, we point out the following conclusions:

- The Darboux transformation in our problem must be carefully approached; the case $B(x)=1$ may indeed lead to difficulties which are linked to the non-revertibility character of the transformation; bringing more complications in the renormalization of the original wavefunctions in quantum mechanics for example.

- The case $B=0$ appears to be more appropriate for practical applications. The case turns out to be identical to the T transformation method which itself orginates from the theorem of separation, so that for a real parameter a, revertibility of this transformation is always guarranteed.
- Extension of this theorem is indeed possible with the 'differential constraint' or with the case of non-symmetric coupling which can be solved by use of the $N(X, Y)$ transformation.
- Finally, the combination of the T and N approaches enables an extension to the $n>2$ case.

It is expected that these methods will become useful tools in the mathematical construction of physical models involving many coupled states.

Acknowledgments

I thank B N Zakhariev (Dubna), F Cannata (Bologna) and V P Berezovoj (Kharkov) for useful correspondence, and the referees of this journal for constructive comments.

References

Amado R D, Cannata F and Delonder J P 1988 *Phys. Rev. Lett.* **61** 2901; *Phys. Rev.* A **38** 3797
Bougouffa S and Cao x C 1988 *J. Physique CI* no 3 **49** 251
—— 1989 *J. Physique CI* no 1 **50** 119
Cao x C 1981 *J. Phys. A: Math. Gen.* **14** 1069
—— 1982 *J. Phys. A: Math. Gen.* **15** 2727
—— 1988 *J. Phys. A: Math. Gen.* **21** 617
—— 1991 *J. Phys. A: Math. Gen.* **24** L1155–65
Lamb J L Jr 1980 *Elements of Soliton Theory* (New York: Wiley) p 243
Mayer H 1985 *J. Phys. A: Math. Gen.* **18** 1085
—— 1988 *J. Phys. A: Math. Gen.* **21** 2075
Lane N F and Lin C C 1964 *Phys. Rev.* A **133** 947
Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
Quesne C 1991 *Int. J. Mod. Phys.* A **6** 1567
Seaton M J 1955 *Proc. Phys. Soc.* **68** 457
—— 1961 *Proc. Phys. Soc.* **77** 174
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** L697
Ui H 1984 *Prog. Theor. Phys.* **72** 192, 813
Witten E 1981 *Nuc. Phys.* B **185** 573